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LETTER TO THE EDITOR

The fractal dimension of the minimum path in two- and three-dimensional percolation

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Abstract. We calculate the fractal dimension d_{min} of the shortest path l between two points on a percolation cluster, where $l \sim r^{d_{min}}$ and r is the Pythagorean distance between the points. We find $d_{min} = 1.130 \pm 0.002$ for $d = 2$ and 1.34 ± 0.01 for $d = 3$.

What is the length l of the shortest path or 'chemical distance' between two points of a random material? In general, l is greater than r, the Pythagorean distance between the points. If the object is self-similar ('fractal') on length scales $r < \xi$ (where ξ is the pair connectedness length), then by definition the density ρ decreases as the size increases as $\rho \sim r^{d_f - d}$, where d_f is the fractal dimension. That $d_f - d$ is negative implies that when a fractal is examined on larger and larger scales there must occur 'holes' on larger and larger scales, up to the size of the connectedness length ξ.

Now as r increases, l increases faster since larger and larger 'holes' must be circumnavigated by the shortest path. Previous work suggests that

$$l \sim r^{d_{min}} \qquad r < \xi \tag{1}$$

where d_{min}, the fractal dimension of the shortest path, is bounded from below by unity. Most studies agree that d_{min} is universal in the sense that it depends only on the dimension d of the underlying lattice and not on the type of lattice or other details of the problem.

One particularly well studied fractal object is the incipient infinite cluster in percolation. However, most studies have been limited to $d = 2$ lattices, and even for the case $d = 2$ different studies disagree on the numerical value for d_{min}. A precise value of d_{min} is important in many physical applications of this concept. For example, suppose we consider a forest fire burning in a fractal landscape. The velocity of the moving front of the fire scales in time with the exponent $d_{min} - 1$; since d_{min} is numerically close to one if $d = 2$ even a small error in d_{min} can have a large effect on the velocity exponent. Apart from the practical applications of d_{min}, this exponent is of considerable theoretical importance since it is the analogue in percolation of the dynamic scaling exponent z of critical point phenomena—it governs the fashion in which information is 'propagated' in time from one point to another. Based on this analogy, one might expect that d_{min} is not related to other percolation exponents, just as z is not related to other critical point exponents. Nonetheless, an intriguing conjecture has been made (Havlin and Nossal 1984) relating d_{min} to the fractal dimension d_{red} of the singly connected 'red' bonds (Stanley 1977, Coniglio 1981) and the fractal dimension d_{f} of the percolation cluster

$$d_{\min} = d_{\text{f}} - d_{\text{red}} = 55/48 = 1.146. \tag{2}$$

The critical exponents of many $d = 2$ systems are now known exactly, so a highly accurate estimate of $d_{\min}$ could be useful to test the accuracy of conjectures such as (2) and also to motivate attempts to calculate $d_{\min}$ exactly.

For reasons such as these, we have undertaken a rather large-scale simulation effort designed to obtain accurate numerical estimates for the dynamic exponent $d_{\min}$ for both $d = 2$ and $d = 3$. To accomplish this, we have systematically studied a sequence of finite-size systems increasing from typically a few hundred sites to almost two million sites. For each system, we have obtained statistics of up to 50 000 distinct realisations. This effort has consumed approximately 10 000 hours of CPU time on an IBM 3091 mainframe computer.

Our procedure is as follows. For a fixed box size of edge b we first generate a percolation cluster at the percolation threshold p_c, which we take to be 0.592 77 for $d = 2$ (Gebele 1984, Ziff and Sapoval 1986) and 0.3117 for $d = 3$ (Heermann and Stauffer 1981). Next we identify the largest cluster, and we move two bus bars from two diagonally opposed corners of the box toward the centre until we contact the largest cluster. The Pythagorean distance r between the two points at which the cluster touches the two bus bars is recorded. Then we calculate the minimum path length between the two points by 'burning' the cluster (Herrmann *et al* 1984). Starting at one of the two sites at each burning step all the nearest neighbours are burned that belong to the cluster and have not yet been burned. This is done until after l burning steps the point on the opposite bus bar is burned.

For each system site b we calculate the averages of the Euclidean distance $\langle r\rangle_b$ and of the chemical distance $\langle l\rangle_b$ and also over various moments of l:

$$(m_2)^2 = \langle l^2\rangle_b - \langle l\rangle_b^2$$

$$(m_4)^4 = \langle l^4\rangle_b - 4\langle l^3\rangle_b\langle l\rangle_b + 6\langle l^2\rangle_b\langle l\rangle_b^2 - 3\langle l\rangle_b^4$$

$$(m_6)^6 = \langle l^6\rangle_b - 6\langle l^5\rangle_b\langle l\rangle_b + 15\langle l^4\rangle_b\langle l\rangle_b^2 - 20\langle l^3\rangle_b\langle l\rangle_b^3 + 15\langle l^2\rangle_b\langle l\rangle_b^4 - 5\langle l\rangle_b^6$$

$$\begin{aligned}(m_8)^8 = \langle l^8\rangle_b &- 8\langle l^7\rangle_b\langle l\rangle_b + 28\langle l^6\rangle_b\langle l\rangle_b^2 \\ &- 56\langle l^5\rangle_b\langle l\rangle_b^3 + 70\langle l^4\rangle_b\langle l\rangle_b^4 - 56\langle l^3\rangle_b\langle l\rangle_b^5 + 28\langle l^2\rangle_b\langle l\rangle_b^6 - 7\langle l\rangle_b^8.\end{aligned}$$

In figure 1 we show how these quantities behave as a function of b for $d = 2$ and 3. We see that the points fall quite well on straight lines even for the higher moments in the log-log plot. Clearly $\langle r\rangle_b \propto b$ as expected. Within our error bars we find $m_6 \propto m_4 \propto m_2 \propto \langle l\rangle_b \propto b^{d_{\min}}$ in $d = 2$ and 3 which means that there is gap scaling and apparently no multifractality. The slopes yield $d_{\min} = 1.135 \pm 0.005$ in $d = 2$ and $d_{\min} = 1.33 \pm 0.03$ in $d = 3$.

We note that in figure 1 corrections to scaling to $\langle l\rangle_b$ and its moments are virtually undetectable since already for the smallest systems the points already follow the asymptotic behaviour within the statistical error bars. This is interestingly different if one plots $\langle l\rangle_b$ against $\langle r\rangle_b$ where a curvature in the data is apparent. One sees actually in figure 1 that there are some deviations of $\langle r\rangle_b$ from the straight line for small b.

Another way of analysing the data is the ratio method, i.e. obtaining for each pair of successive values of b the local slope in figure 1 which gives an effective $d_{\min}(b)$. In figure 2 we plot $d_{\min}(b)$ as a function of b^{-1}. We see that the data are not independent

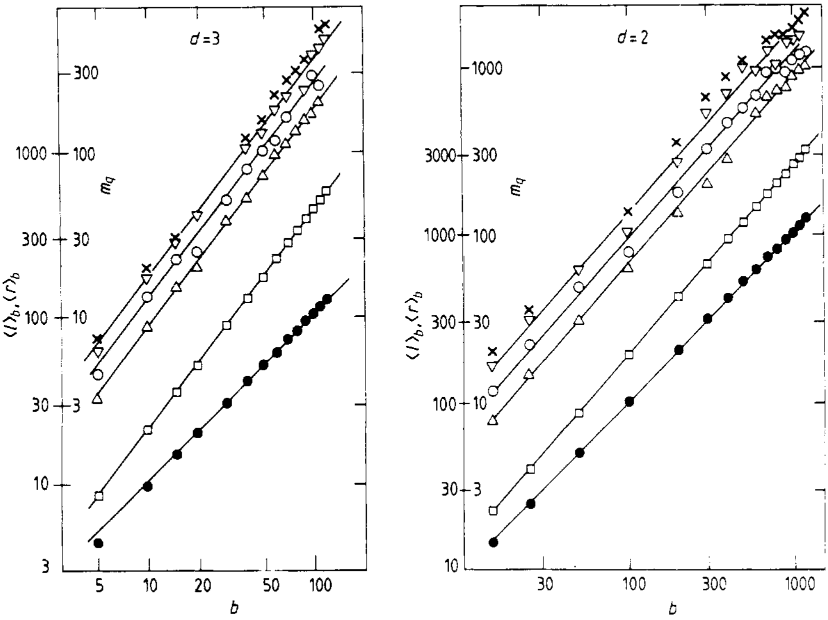

Figure 1. $\langle l\rangle_b$ (●), $\langle r\rangle_b$ (□) and moments m_q as a function of b in $d=2$ and $d=3$. The moments are $q=2$ (△), $q=4$ (○), $q=6$ (▽), and $q=8$ (×).

of b so that there is some correction to scaling. Extrapolating to $b\to\infty$ yields from figure 2: $d_{\rm min}=1.130\pm0.005$ in $d=2$ and $d_{\rm min}=1.38\pm0.04$ in $d=3$. Due to the strong statistical fluctuations at large b it is not possible to obtain the correction to scaling exponent.

One way to suppress the strong fluctuations is to calculate another effective $d_{\rm min}$ in which one calculates in figure 1 the slope via a linear regression without taking into account sizes less than a cutoff size b^*. In figure 3 we show $d^{\rm e}_{\rm min}$ as a function of the inverse of b^*. For $d=3$ we extrapolate $d_{\rm min}=1.338\pm0.004$. In $d=2$ the data have no clear trend and statistical fluctuations dominate; a value of $d_{\rm min}=1.129\pm0.001$ seems plausible.

Edwards and Kerstein (1985) (see also Kerstein and Edwards 1986) suggested the possibility of $d_{\rm min}=1$ in $d=2$ but with logarithmic corrections to scaling in b of the type $(\ln b)^{-1}$. The increase (instead of decrease) of the effective $d_{\rm min}(b)$ for $d=2$ in figure 2 when $b\to\infty$ already rules out a limiting value of 1 regardless of the type of corrections. We analysed our data, however, also for a behaviour $\langle l\rangle_b\sim\langle r\rangle_b(\ln\langle r\rangle_b)^x$ but the data do not fall at all on a straight line in a log-log plot so this possibility can be ruled out.

Instead of averaging together all data from a given box size b, we also considered all pairs (r, l) regardless of the box they came from as independent data points and used linear regression in the log-log plot on the whole cloud of data. We also considered the linear regression of the cloud taking away data for small values of l or r for different cutoffs. We also binned the data according to l and according to r, calculated averages $\langle r\rangle_l$ and $\langle l\rangle_r$ respectively and tried to get $d_{\rm min}$ from the relation between l and $\langle r\rangle_l$ and between $\langle l\rangle_r$ and r. None of these many ways of getting $d_{\rm min}$ gave good results: the values fluctuated very much, strongly dependent on the cutoffs and even dependent

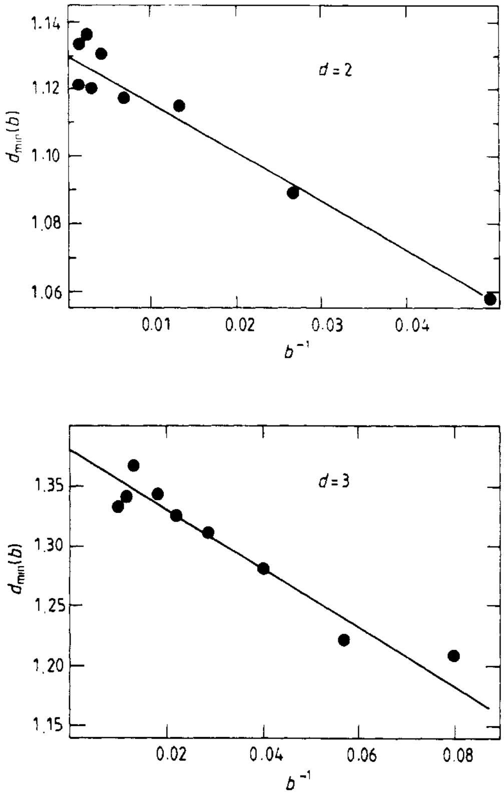

Figure 2. Effective $d_{min}(b)$ as a function of b^{-1} for $d=2$ and $d=3$. The straight line is a guide to the eye.

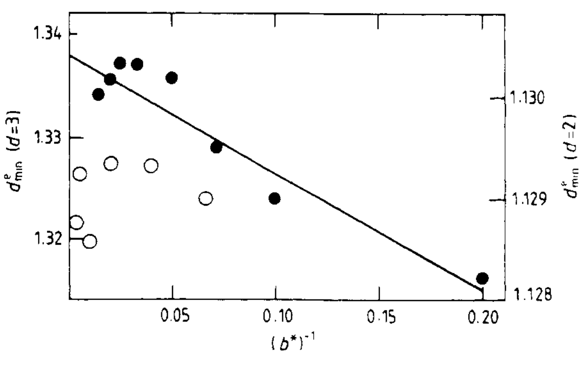

Figure 3. Effective d^e_{min} as a function of the cutoff b^{*-1} for $d=2$ (○) and $d=3$ (●). The straight line is a guide to the eye.

on the detailed linear regression formula (formula obtained through minimising y coordinates or formula obtained by minimising the Euclidean distances). These effects are due to the boundaries of our boxes which systematically neglect paths of a certain type, namely those that would go out of the shape of the square given by the box. The best way to control this effect is therefore by comparing results of different box sizes with each other, i.e. what we previously did in figures 1-3.

Summarising our results we obtain the exponents $d_{min} = 1.130 \pm 0.002$ for $d = 2$ and $d_{min} = 1.34 \pm 0.01$ for $d = 3$. They are consistent with but more precise than the previous numerical estimates in $d = 2$ of Pike and Stanley (1981), Havlin and Nossal (1984), Herrmann *et al* (1984) and Grassberger (1985), and in $d = 3$ of Alexandrowicz (1980), Herrmann *et al* (1984) and Grassberger (1986a, b). In $d = 2$ we can exclude a recent conjecture $d_{min} = 17/16$ made by Larsson (1987) and we can also exclude the $d_{min} = d_f - d_{red}$ of Havlin and Nossal (1984). The later value has been known to be inconsistent with some numerical work and ε expansion (for a discussion see, e.g., Grassberger (1986a, b)). We propose, however, the relation $d_{min} = 2 - d_B + d_{red}$ which is in good agreement with our data, where d_B is the fractal dimension of the backbone (Herrmann and Stanley 1984).

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